# WICK TYPE SYMBOL AND DEFORMED ALGEBRA OF EXTERIOR FORMS

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The covariant descripion is constructed for the Wick-type symbols on symplectic manifolds by means of the Fedosov procedure. The geometry of the manifolds admitting this symbol is explored. The superextended version of the Wick-type star-product is introduced and a possible application of the construction to the noncommutative field theory is discussed.

## 1. **Introduction**

The aim of the paper is twofold. First we construct covariant Wick-type star-product within the framework of the Fedosov deformation quantization [1]. Second we propose a canonical superextension of the construction along the lines of the Bordemann approach [2] and present a Wick-type deformation for the exterior algebra of the base even manifold. The first step is motivated by a wide physical application of the Wick-type star-product especially for the models with the infinite degrees of freedom. The second one is motivated by noncommutative field theory [8, 9, 10] on a curved manifold, where the covariant definition is required for the deformed product between the *tensor observables*. We also hope that recently found relationship between the Fedosov deformation and BRST theory [11], being combined to the Wick type symbol technique, should pave the way to studying anomalies in a wide class of the filed theory problems making use of the deformation quantisation tools.

## 2. **The Wick-type symbol in the deformation quantization**

Under the term "Wick-type" we understand a broad class of symbols incorporating, along with the ordinary (genuine) Wick symbols, the so-called $qp$-symbols as well as various mixed possibilities commonly regarded as the pseudo-Wick symbols. To give a more precise definition of what is meant here consider first the linear symplectic manifold $\mathbb{R}^{2n}$ equipped with the canonical Poisson brackets $\{y^i, y^j\} = \omega^{ij}$. Then the usual Weyl-Moyal product of two observables, defined as

$$a * b(y) = \exp\left(\frac{i\hbar}{2}\omega^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right) a(y)b(z)|_{z=y},$$

turns the space of smooth functions in $y$ to the non-commutative associative algebra with unit called the algebra of Weyl symbols. The transition from the Weyl to Wick-type symbols is achieved by adding a certain complex-valued symmetric tensor $g$ to the Poisson one $\omega$ in the formula for the Weyl-Moyal $*-$product,

$$a *_g b(y) = \exp\left(\frac{i\hbar}{2}\Lambda^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right) a(y)b(z)|_{z=y}, \tag{1}$$

$$\Lambda^{ij} = \omega^{ij} + g^{ij}, \qquad \Lambda^\dagger = -\Lambda.$$

Although the associativity of the modified product holds for any constant $g$ the Wick type symbols are extracted by the additional half-rank condition $rank\Lambda = n$. In particular, the genuine Wick symbol corresponds to a pure imaginary $g$ while the real $\Lambda$ is associated with *qp*-symbol. In the general case one can easily show that the dual complexified phase space $\mathbb{C}^{2n}$ is splitted into a direct sum of two transverse Lagrangean subspaces, which are left and right kernel subspaces for the matrix $\Lambda^{ij}$.

Turning to the curved manifold $M$ we just replace the constant matrix $\Lambda^{ij}$ by a general complex-valued bilinear form $\Lambda^{ij}(x) = \omega^{ij}(x) + g^{ij}(x)$ with the antisymmetrical part being the non-degenerate Poisson tensor. Then the Wick-type star-product is defined to satisfy the following "boundary condition"

$$a * b = ab - \frac{i\hbar}{2}\Lambda^{ij}\partial_i a \partial_j b + \dots \tag{2}$$

$\hbar$ being the formal deformation parameter ("Plank constant"), and dots mean the terms of higher orders in $\hbar$. As in a general case of the symplectic manifold the condition (2) is compatible with so-called the *correspondence principle* of quantum mechanics:

$$\lim_{\hbar\to 0}\frac{i}{\hbar}(a * b - b * a) = \{a, b\}, \tag{3}$$

where $\{\cdot,\cdot\}$ means the Poisson bracket associated to $\omega^{ij}$. It turns out that if there exists a torsion-free linear connection $\nabla$ preserving $\Lambda^{ij}$ the star-product satisfying the condition (2) can be achieved by a minimal modification of the Fedosov method. Namely, if we replace the $\circ$-multiplication in the Weyl algebra bundle [1] by the that of the Wick type (1) all the steps and the theorems of the Fedosov method is generalized in a straightforward manner and thus we get the star-product of the Wick type.

The only crucial point of this program is the existence of a torsion-free linear connection $\nabla$ preserving $\Lambda$. One may readily find that the necessary and sufficient condition for such a connection to exist is the integrability of the right and left kernel distribution of $\Lambda(x)$. When the latter condition is fulfilled $\nabla$ is just the Levi-Civita connection associated to the symmetric and non-degenerate form $g(x)$ and the right and left kernel distributions define the transverse polarizations of the symplectic manifold $(M, \omega)$. Thus we see that the symplectic manifold admitting the Wick-type star-product are necessarily equipped with the pair of the transverse polarizations.

After the paper [3], the symplectic manifold equipped by a torsion-free symplectic connection is usually called as Fedosov manifold because precisely these data – symplectic structure and connection – enter to Fedosov's star-product. The Wick deformation quantization (as we have defined) involves one more geometric structure - a pair of transverse polarizations, and, by analogy to the previous case, the underlying manifold may be called as the *Fedosov manifolds of Wick type* or *FW-manifold for short*. The extended list of examples of the FW-manifolds is provided by the Kähler manifolds. It is the case when the matrix of the tensor $\Lambda^{ij}$ is the anti-Hermitian in local real coordinates. There are also examples of the FW-manifolds having no Kähler structure but assigned instead by two real transverse polarizations. The last situation is strikingly illustrated by the one-sheet hyperboloid embedded into three dimensional Minkowski space as the surface

$$x^2 + y^2 - z^2 = 1$$

$x, y, z$ being linear coordinates in $R^{2,1}$. one ($\det g < 0$). The integral leaves of the corresponding real polarizations coincide here with two transverse sets of linear generatings of

hyperboloid's surface being, in turn, the isotropic geodesics of the respective metric structure.

At present there is a large amount of literature concerning the deformation quantization on polarized symplectic manifolds (see i.g. [5],[6],[7] and references therein) beginning with the pioneering paper by Berezin [4] on the quantization in complex symmetric spaces. However, in all the papers cited above the very definition and the construction of the Wick-type star-product are based on the explicit use of local coordinates adapted to the polarization (or *separation of variables* in terminology of work [7]).

## 3. Superextension

Now we present construction of the double dimensional superextension of the FW-manifold $M$ and perform its deformation quantization. We construct the supermanifold supplying the initial FW-manifold by additional odd variables $\theta^i$, where $i$ is the index of the tangent space. Then the formal series

$$\mathcal{C} \ni a = a(x,\theta) = \sum_{k=1}^{2^{2n}} a(x)_{i_1\ldots i_k}\theta^{i_1}\ldots\theta^{i_k}, \tag{4}$$

where the coefficients $a(x)_{i_1\ldots i_k}$ are antisymmetric with respect to the index permutations are thought to be superfunctions constituting a supercommutative superalgebra, which we will denote by $\mathcal{C}$ . As it follows from the results by Bordemann [2] the Grassmann multiplication in $\mathcal{C}$ can be formally deformed into associative $\mathbb{Z}_2$-graded multiplication $*$ on $\mathcal{C}[[\hbar]]$ whenever the symplectic and the metric structures are given on the base even manifold. The remarkable feature of this construction is that the deformation quantization for the algebra of super functions is performed first and the super Poisson bracket arises here *a posteriori* as $\hbar$-linear term of the supercommutator.

Here we modify Bordemann's approach to the quantization of the supermanifold [2] and present the deformation quantization of the Wick type for the superextension of the FW-manifold. Since the basic steps of our construction are essentially similar to those in the original papers by Fedosov [1] and Bordemann [2] we consider them very briefly dwelling only upon peculiarities.

First we define a superextension of the Weyl algebra bundle $\mathcal{W}$, introduced in [1].

**Definition 1.** *The bi-graded superalgebra* $\mathcal{SA} = \oplus_{m,n=0}^{\infty}(\mathcal{SA})_{m,n}$ *with unit over* $\mathbb{C}$ is *a space of formal series,*

$$a(x,\theta,y,dx,\hbar) = \sum_{2k+p\geq 0} \hbar^k a_{k\, i_1\ldots i_p j_1\ldots j_{q'} l_1\ldots l_{q''}}(x) y^{i_1}\ldots y^{i_p}\theta^{j_1}\ldots\theta^{j_{q'}}dx^{l_1}\ldots dx^{l_{q''}}, \tag{5}$$

*multiplied with the help of associative* $\circ$*-product of the form*

$$a\circ b = \tag{6}$$

$$= exp\frac{i\hbar}{2}\Lambda^{ij}(x)\left(\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j} + \frac{\overleftarrow{\partial}}{\partial\theta^i}\frac{\overrightarrow{\partial}}{\partial\chi^j}\right) a(x,y,\theta,dx,\hbar)b(x,z,\chi,dx,\hbar)|_{y=z,\theta=\chi}$$

*A general term of the series (5) is assigned by the bi-degree* $(2k+p+q', q'')$.

The expansion coefficients $a_{k\, i_1 \ldots i_p j_1 \ldots j_{q'} l_1 \ldots l_{q''}}(x)$ are considered to be components of covariant tensors on $M$ symmetric in $i_1, \ldots, i_p$ and antisymmetric with respect to $j_1, \ldots, j_{q'}$ and $l_1, \ldots, l_{q''}$. Note that we omit the sign $\wedge$ between the differentials regarding $dx^i$ and $\theta^j$ as the set of $2n$ anti-commuting variables:

$$dx^i dx^j = -dx^j dx^i, \quad \theta^i \theta^j = -\theta^j \theta^i, \quad dx^i \theta^j = -\theta^j dx^i \tag{7}$$

Hence, $\mathcal{SA}$ is also $\mathbb{Z}_2$-graded algebra with respect to the Grassmann parity $q = q' + q''$ (5) and the supercommutator of two homogeneous elements $a, b \in \mathcal{SA}$ with the parities $q_1$and $q_2$ is defined as

$$[a, b] = a \circ b - (-1)^{q_1 q_2} b \circ a \tag{8}$$

Let us introduce the nilpotent operator $\delta = dx^i \frac{\partial}{\partial y^i}$ which will be rather important in formulating the respective analogues of the Fedosov equation. It is easily seen to be the inner superderivation of the superalgebra $\mathcal{SA}$.

The nontrivial cohomology of $\delta$ coincides with the space of superobservables $\mathcal{C}[[\hbar]]$ and the operator $\delta^{-1}$

$$\delta^{-1} a = y^k i\left(\frac{\partial}{\partial x^k}\right) \int\limits_0^1 a(x, ty, tdx, \hbar) \frac{dt}{t}, \tag{9}$$

is the partial homotopy operator for $\delta$ in the sense of "Hodge-De Rham" decomposition

$$a = \sigma(a) + \delta\delta^{-1} a + \delta^{-1}\delta a\,, \tag{10}$$

where $\sigma(a) = a(x, \theta, 0, \hbar)$ and the interior derivation $i\left(\frac{\partial}{\partial x^k}\right)$ acts on the forms by the rule

$$i\left(\frac{\partial}{\partial x^k}\right) a_{i_1 \ldots i_m} dx^{i_1} \ldots dx^{i_m} = m a_{k i_2 \ldots i_m} dx^{i_2} \ldots dx^{i_m}$$

The covariant derivative $\nabla$ given on the base FW-manifold induces the superderivation in $\mathcal{SA}$

$$\nabla : (\mathcal{SA})_{n,m} \to (\mathcal{SA})_{n,m+1}, \tag{11}$$

$$\nabla = dx^i \left( \frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij}(x) \frac{\partial}{\partial y^k} - \theta^j \Gamma^k_{ij}(x) \frac{\vec{\partial}}{\partial \theta^k} \right),$$

$\Gamma^k_{ij}$ are Christoffel symbols of the connection associated to $\Lambda$. It is easy to check that $\nabla$ anti-commutes with $\delta$ and its curvature is given by

$$\nabla^2 a = \nabla(\nabla a) = \frac{1}{i\hbar}[R\,, \,a], \tag{12}$$

$$R = \frac{1}{4} R_{ijkl} y^i y^j dx^k dx^l + \frac{1}{4} \mathcal{R}_{ijkl} \theta^i \theta^j dx^k dx^l \tag{13}$$

where we have used the notations $R_{ijkl} = \omega_{im} R^m_{\ jkl}$ and $\mathcal{R}_{ijkl} = g_{im} R^m_{\ jkl}$. Following Fedosov method, one can combine $\nabla$ with an inner derivative to get the Abelian connection of the form

$$D = \nabla - \delta + \frac{1}{i\hbar}[r, \cdot] = \nabla + \frac{1}{i\hbar}[\omega_{ij} y^i dx^j + r, \cdot], \quad r = r_i(x, \theta\,, y, \hbar) dx^i, \tag{14}$$

Denote $\mathcal{SA}_D = \ker D$. The next assertions is the super counterpart of that stated in Fedosov's original theorems [1, Theorems 3.2, 3.3].

**Theorem 1.** *With the above definitions and notations we have:*

*i) there is a unique Abelian connection D (14) for which*

$$\delta^{-1} r = 0, \quad r_i(x, \theta\,, 0, \hbar) = 0$$

*and r consists of monomials whose first degree is no less then 3;*

*ii) $\mathcal{SA}_D$ is a subalgebra of $\mathcal{SA}$ and the map $\sigma$ being restricted to $\mathcal{SA}_D$ defines a linear bijection onto $\mathcal{C}[[\hbar]]$;*

**Corollary 1**. *The pull-back of $\circ$-product via $\sigma^{-1}$ induce an associative $*$-product on the space of physical observables $\mathcal{C}[[\hbar]]$, namely*

$$a * b = \sigma(\sigma^{-1}(a) \circ \sigma^{-1}(b)) \tag{15}$$

**Proof** may be directly read off from [2, Theorems 2.1,2.2].

The peculiar property of the presented Wick-type deformation is that the superalgebras $(\mathcal{C}[[\hbar]], *)$ and $(C^\infty(M)[[\hbar]], *')$ , where $*'$ is the respective Wick-type star-product on the base (even) manifold are related to each other in the sense of the following proposition

**Proposition 1.** *Let $\pi : \mathcal{C}[[\hbar]] \to C^\infty(M)[[\hbar]]$ be the canonical projection defined as $\pi a(x, y, \hbar, \theta) = a(x\,, y\,, \hbar, 0)$. Consider the algebras $\mathcal{C}[[\hbar]]$ and $C^\infty(M)[[\hbar]]$ as left(right) moduli over $C^\infty(M)[[\hbar]]$. Then $\pi$ defines a homomorphism of the introduced moduli, that is for $\forall a \in C^\infty(M)[[\hbar]]$ and $\forall b \in \mathcal{C}[[\hbar]]$ we have:*

$$a *' (\pi b) = \pi(a * b), \quad (\pi b) *' a = \pi(b * a) \tag{16}$$

It is easy to check that the equations (16) do not hold for the case of the Weyl star-product obtained for example by Bordemann's approach [2].

## 4. **Conclusion**

In this paper we have realized the Wick-type star-product following to the Fedosov approach to the deformation quantization. We have observed that the most natural geometrical framework for this star-product is the so-called Fedosov manifold of Wick type. This is defined as the real symplectic manifold equipped with a complex-valued half-rank bilinear form $\Lambda$ preserved by some torsion-free connection and it is shown to be necessarily the manifold admitting two transverse polarizations.

We have shown that any FW-manifold can be canonically extended to the doubled dimensional supersymplectic manifold associated to the tangent bundle of the initial manifold. Moreover, the latter can be easily then quantized along the line of the Fedosov approach and this quantization procedure, yields the natural definition of the deformed product between antisymmetrical covariant tensors on the base even manifold. The latter is due to

the natural one-to-one correspondence between the elements of the exterior algebra and the superfunctions (4) of the introduced supermanifold. This correspondence can be easily restored by its definition on the monomial elements of the exterior algebra

$$\omega_{i_1\dots i_p}dx^{i_1}\dots dx^{i_p}\mapsto\omega=\omega_{i_1\dots i_p}\theta^{i_1}\dots\theta^{i_p}$$

We hope that this deformed associative product may find its application in the noncommutative field theory in particular for constructing the noncommutative Yang-Mills theory [8, 9, 10] on the curved symplectic manifolds. The problem of realizing the noncommutative Yang-Mills theory on a curved manifold is firstly posed in the paper by Connes, Douglas and Schwarz [9] and at first sight it seems to have a natural solution in the context of the deformation quantization. However the deformation quantization poses the problem of constructing the deformed algebra of scalar functions on the symplectic manifold. Hence it is insufficient for constructing the noncommutative Yang-Mills theory, which should describe the dynamics of tensor fields as well. Thus one firstly has to realize the star-product between tensor observables on the symplectic manifold. The star-product in addition should be defined in a generally covariant way since otherwise it would be impossible to present the invariant action for the theory in question. The quantization of the supermanifold presented above suggests a possible solution for the problem as it yields the definition for the covariant deformation of the exterior algebra on the initial even manifold. However the deformed algebra of the forms cannot be restricted for example to the set of the 1-forms only thus a hypothetical noncommutative Yang-Mills theory would describe the dynamics of the forms of all the possible ranks.

Another ingredient which is important for constructing the action for the noncommutative Yang-Mills theory is the metric structure. This structure, in turn, being naively inserted into the action functional may violate the gauge invariance of the theory even in the case of the ordinary Weyl-Moyal star-product. Thus the metric and symplectic structures should be considered on equal foot just like in the construction of the Wick-type star-product. Finally the construction of the gauge invariant action requires the definition of the trace measure for the aforementioned superalgebra. We will consider this problem elsewhere.

**Acknowledgments.** It is our pleasure to express our sincere gratitude to I. Batalin, I. Gorbunov, M. Grigoriev, M. Henneaux A. Karabegov, R.Marnelius, A. Nersessian and M.Vasiliev for useful discussions related to various aspects of this paper. This work has been partially supported by RFBR grant 00-02-17-956.

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